Birational Geometry Seminar, Week 38
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On the cone conjecture for log Calabi-Yau threefolds

Morrison '93 (conj.)

X : CY 3 $\quad \begin{cases} X : \text{cx. proj. var. of dim 3} \\ K_X = 0 \\ \pi_1(X) = 0 \end{cases}$

$\mathrm{Nef}(X) \subset H^2(X, \mathbb{R})$.

Then (i) $\mathrm{Aut}(X) \curvearrowright \mathrm{Nef}(X)$ with a rational polyhedral fundamental domain (R.P.F.D.)

(ii) $\mathrm{PsAut}(X) \curvearrowright \mathrm{Mov}(X)$ with an R.P.F.D.

$X \dashrightarrow X$ isom. outside
a codim 2 subset of
domain and codomain

In particular,

$\mathrm{Aut}(X) \curvearrowright \{\text{faces of } \mathrm{Nef}(X)\}$ with finitely many orbits;

$\mathrm{PsAut}(X) \curvearrowright \{\text{faces of } \mathrm{Mov}(X)\}$ " ".

Y : sm. proj. 3-fold

$\mathrm{Nef}^e(Y) = \mathrm{Eff}(Y) \cap \mathrm{Nef}(Y)$

$\mathrm{Mov}^e(Y) = \mathrm{Eff}(Y) \cap \mathrm{Mov}(Y)$

<u>Important Idea</u>

If Z is a flop of Y, then the extremal rays of $\mathrm{Curv}(Z)$ in the $K_Z < 0$ region can be either

Type (1) : blowup of a smooth curve Γ

$$Z \longrightarrow Z' \supseteq \Gamma$$

or Type (6) : conic bundle

Mori's classification of extremal rays of $\overline{\text{Curv}}(Y)$ (Kollár-Mori, Thm 1.32, p. 28)

X: nonsingular projective 3-fold over $\mathbb{C}$

$\text{cont}_R : Y \to X$ contraction of a K_Y-negative extremal ray $R \subset \overline{\text{Curv}}(Y)$

Then we have the following possibilities:

(1) cont_R is the (inverse of) the blowup of a smooth curve in the smooth 3-fold Y.

(2) cont_R is the (inverse of) the blowup of a smooth point of the smooth 3-fold Y.

(3) cont_R is the (inverse of) the blowup of a point of Y that is locally analytically given by $x^2 + y^2 + z^2 + w = 0$.

(4) cont_R is the (inverse of) the blowup of a point of Y that is locally analytically given by $x^2+y^2+z^2+w^3=0$.

(5) cont_R contracts a smooth $\mathbb{CP}^2$ with normal bundle $\mathcal{O}(-2)$ to a point of multiplicity 4 on Y, which is locally analytically the quotient of $\mathbb{C}^3$ by the involution

$$(x,y,z) \mapsto (-x,-y,-z).$$

(6) $\dim(Y) = 2$ and cont_R is a fibration whose fibers are plane conics (general fibers are smooth)

(7) $\dim(Y) = 1$ and the general fibers are del Pezzo surfaces.

(8) $\dim(Y) = 0$ and $-K_X$ is ample, so X is a Fano variety.

Main Theorem

Y: sm. proj. 3 fold admitting a K3 fibration $\begin{array}{c} Y \\ \downarrow f \\ \mathbb{P}^1 \end{array}$ s.t. $-K_Y = f^*\mathcal{O}(1)$

Then $\mathrm{PsAut}(Y)$ acts on

(1) the Type (6) codimension-one faces of $\mathrm{Mov}^e(Y)$ with finitely many orbits;

(2) the Type (1) codim.-1 faces of $\mathrm{Mov}^e(Y)$ with finitely many orbits if $H^3(Y,\mathbb{C}) = 0$.

Remark If $H^3(Y,\mathbb{C}) = 0$, then in the case of Type (1) (blowup of smooth curve T), the genus $g(T) = 0$.

This takes care of $K<0$ region.

Conjecture $\mathrm{PsAut}(Y) \curvearrowright \{\text{Type (1) codim 1 faces of } \mathrm{Mov}^e(Y)\}$ w/ finitely many orbits.

Thm (Kawamata) $\mathrm{PsAut}(Y) \curvearrowright \{\text{codim } 1 \text{ faces of } \mathrm{Mov}(Y) \text{ containing } -K_Y\}$ w/ finitely many orbits.

Thm Conj (assume true) + Kawamata's Thm give:

$\mathrm{PsAut}(Y) \curvearrowright \{\text{codim } 1 \text{ faces of } \mathrm{Mov}^e(Y)\}$ with finitely many orbits.

Remark This statement is implied by the Kawamata-Morrison-Totaro (KMT) cone conj:

KMT cone conj

(Y, Δ): klt — "Kawamata log terminal"

e.g. Y sm., $\Delta = \sum a_i \Delta_i \subset Y$ n.c.d. with $a_i < 1$

Then (Y, Δ) klt

and $K_Y + \Delta = 0$ — "klt log CY"

Then (i) $\mathrm{Aut}(Y, \Delta) \curvearrowright \mathrm{Nef}^e(Y)$ with RPFD; and

(ii) $\mathrm{PsAut}(Y, \Delta) \curvearrowright \mathrm{Mov}^e(Y)$ with PPFD.

Pf (sketch of main ideas)

Consider Y : CY3 with a K3 fibration $\begin{array}{c} Y \\ f \downarrow \\ \mathbb{P}^1 \end{array}$ s.t. $-K_Y = f^* \mathcal{O}(1)$

Take F : a general fiber of f

$F_1, F_2 \sim F$ smooth fiber

choose $\Delta = \frac{1}{2} F_1 + \frac{1}{2} F_2$.

Then (Y, Δ) klt log CY.

By a result of Birkar-Cascini-Hacon-McKernan,

$$\mathrm{Mov}(Y) = \bigcup_{\substack{Y \dashrightarrow Z \\ \text{SQMs}}} \mathrm{Nef}(Z)$$ is locally R.P. in $\mathrm{Int}(\mathrm{Mov}(Y))$.

Small-$\mathbb{Q}$-factorial modification $Y \dashrightarrow Z$:

Y, Z : $\mathbb{Q}$ factorial

$Y \dashrightarrow Z$ is a birat'l map, an isom. away from codim 2 subsets

Ex Flops are SQMs.

☆ In our setting, the only SQMs are flops / compositions of flops.

A main idea used in proof :

$$S := \bigcup_{\substack{Y \dashrightarrow Z \\ \text{SQMs}}} \{ \text{codim. 1 faces of } \mathrm{Nef}(Z) \text{ in the boundary of } \mathrm{Mov}(Y) \} \twoheadrightarrow \{ \text{codim 1 faces of } \mathrm{Mov}^e(Y) \}.$$

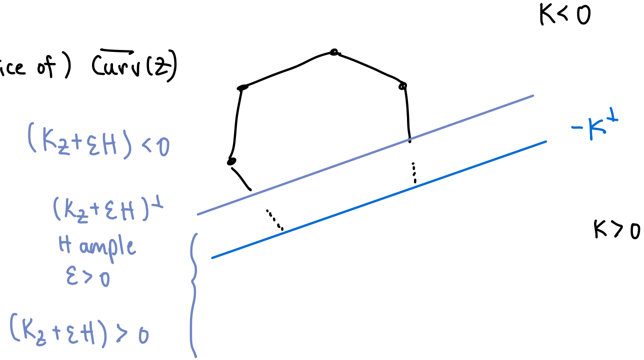
$$\updownarrow$$

$$\bigcup_{\substack{Y \dashrightarrow Z \\ \text{SQMs}}} \{ \text{extremal rays of } \underset{\underset{\mathrm{Nef}(Z)^*}{\shortparallel}}{\overline{\mathrm{Curv}}(Z)} \}$$

(1) Mori's Cone Theorem

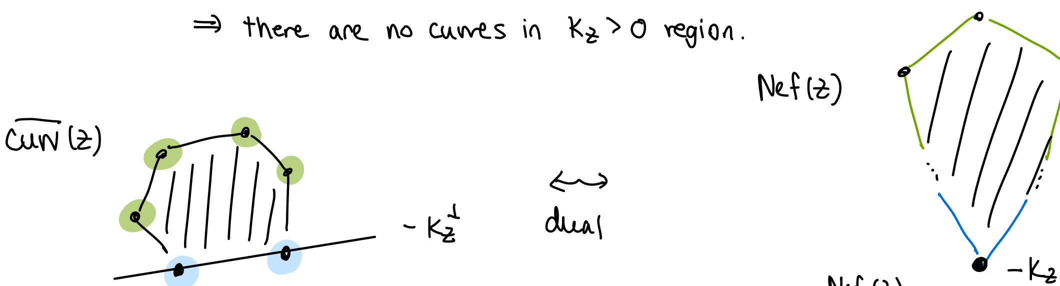
Z : sm. proj. proj.

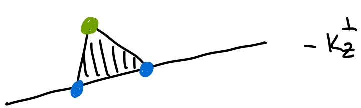
(a slice of) $\overline{\mathrm{Curv}}(Z)$

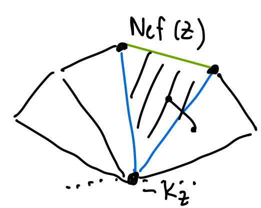


For us, even better: $-K_Z$ (a fiber) is nef

$\Rightarrow$ there are no curves in $K_Z>0$ region.







$\mathrm{Mov}(Y) = \bigcup_{\substack{Y \dashrightarrow Z \\ \text{flop}}} \mathrm{Nef}(Z)$

(2) Mori's classification of $\underbrace{\text{extremal rays of } \overline{\mathrm{Curv}}(Z) \text{ in } K_Z < 0}$

$\Updownarrow$

codim 1 faces of $\mathrm{Mov}(Y)$

<u>Recall</u>

X: nonsing. proj. 3fold $/\mathbb{C}$

$\mathrm{cont}_R : X \to Y$ contraction of a K_X-negative extremal ray $R \subset \mathrm{Curv}(X)$

8 possible types

Eliminate 6

Two remaining possibilities:

<u>Type (1)</u> B.U. of a smooth curve

<u>Type (6)</u> Conic bundle

① describe the green vertices in $\overline{\mathrm{Curv}}(Z)$

② the blue vertices on $-K_Z^{\perp}$ are taken care of by Kawamata

Type (6) Conic bundle corresponding to a ray of $\overline{\mathrm{Curv}}(Z)$.

$$\begin{tikzcd} Y \arrow[r, dashed, "\text{flops}"'] & Z \arrow[r, "g"', "g:\ \text{conic bundle}"] \arrow[d, "f:\ \text{K3 fibration}"] & S \\ & \mathbb{P}^1 & \end{tikzcd}$$

Let $h := g|_F : \underbrace{F}_{\substack{\text{general fiber} \\ \text{(K3 surface)}}} \to S$, finite, degree 2.

- There is an involution i associated to h:
- $S = F/\langle i \rangle$, F : K3
- S is smooth (Mori)

$\Rightarrow$ 2 possibilities: either $\begin{cases} (1)\ \mathrm{Fix}(i) = \text{union smooth curves} \\ (2)\ \mathrm{Fix}(i) = \emptyset \end{cases}$

(1) $\Rightarrow$ S is a rational surface

(2) $\Rightarrow$ S is an Enriques surface ☆ Cone conj. holds

Suppose S is a rational surface. Run MMP on S:

$$\pi : S \longrightarrow \underbrace{\bar{S}}_{\mathbb{P}^2,\ \mathbb{F}_n\ (0 \le n \le 4,\ n \ne 1)}$$

Now we have

$$\begin{tikzcd} F \arrow[r, "h"] \arrow[rr, bend right, "\theta := \pi \circ h"'] & S \arrow[r, "\pi"] & \bar{S} \arrow[r, "\phi"] & \hat{S} \end{tikzcd}$$

and let $L = \theta^* M$, where If $\bar{S} = \mathbb{P}^2$: take $\phi = \mathrm{id}$ and $M = H$ is a hyperplane class

$\Rightarrow L^2 = 2$

If $\bar{S} = \mathbb{F}_n$ $(0 \le n \le 4,\ n \ne 1)$: take $\phi = \mathbb{F}_n \to \underbrace{\mathbb{P}(1,1,n)}_{\hat{S}}$ to be the contraction of the negative section of $\mathbb{F}_n$ and M = the positive section of $\mathbb{F}_n$

$\Rightarrow L^2 = 2M^2 = 2n \le 8$
$\uparrow$
$n \le 4$

<u>Thm</u> (Sterk, 85) F: K3 surface and L: nef line bundle with $L^2 = 2k$ for some fixed $k \in \mathbb{N}$.

Then $\mathrm{Aut}(F) \curvearrowright \{\text{all such } L\}$ with finitely many orbits.

Moreover, we show that the line bundle L gives $F \longrightarrow \hat{S}$. This proves Main Thm (1).

<u>Type (1)</u> B.U. of a smooth curve, or the contraction of a ruled surface E.

So we have a $\mathbb{P}^1$-bundle $g: E \to T$ (and K3 fibr. $f: Y \to \mathbb{P}^1$)

Suppose that $H^3(Y, \mathbb{C}) = 0$. Then genus $g(T) = 0$

$$\Rightarrow T \cong \mathbb{P}^1.$$

In our case, $g: E \to T$ is a trivial $\mathbb{P}^1$-bundle.

$\Rightarrow$ $T \subset F$ for each fiber F of f.

By A.F., $T \subset F$ is a (-2)-curve.

<u>Thm</u> (Sterk) Morrison's cone conj. for K3 surface

Y_η : generic fiber (K3 surface)

Then $\mathrm{Aut}(Y_\eta) \curvearrowright \{(-2)\text{-curves in } Y_\eta\}$ w/ finitely many orbits.

$$\begin{array}{c} \cap\!| \\ \mathrm{PsAut}(Y) \end{array}$$

$\Rightarrow$ $\mathrm{PsAut}(Y) \curvearrowright \{(-2)\text{-curves in } Y_\eta\}$ w/ finitely many orbits

$\uparrow$ $\exists$ injection

$\therefore$ $\mathrm{PsAut}(Y) \curvearrowright \{E \subset Y \mid E : \text{exceptional divisor of Type (1) on } Y\}$ w/ fin. many orbits.

Proves Main Theorem (2).

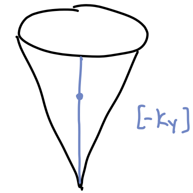
Taken care of $K < 0$.

Thm (Kawamata)

$f : Y \to S$ K3 fibration, $\dim(Y) = 3$

Then $\mathrm{PsAut}(Y/S) \curvearrowright \mathrm{Mov}^e(Y/S)$ with finitely many orbits on faces.

Kawamata's picture lives in $N^1(Y/S) = N^1(Y) \otimes \mathbb{R} / f^* N^1(S) \otimes \mathbb{R}$

$= N^1(Y) \otimes \mathbb{R} / \mathbb{R} \cdot [-K_Y]$ $\supset$ 

for us: $S = \mathbb{P}^1$

so $N^1(S) = \mathbb{Z} \cdot [pt]$

$= \mathbb{Z} \cdot [f^* pt]$

$= \mathbb{Z} \cdot [-K_Y]$

Know: $\mathrm{PsAut}(Y/\mathbb{P}^1) \curvearrowright$ faces of a chamber decomposition containing $[D]$ with finitely many orbits

$\cap$

$\mathrm{PsAut}(Y) \curvearrowright \mathrm{Mov}(Y) = \bigcup_{Y \dashrightarrow Z} \mathrm{Nef}(Z)$ with finitely many orbits.

Main Thm + Kawamata's Thm + Conj $\Rightarrow$ $\mathrm{PsAut}(Y) \curvearrowright \{\text{codim } 1 \text{ faces of } \mathrm{Mov}^e(Y)\}$ w/ finitely many orbits.

<u>Remark</u> There are many examples of Y (sm. proj. 3fold) w/ a K3 fibration $\begin{array}{c} Y \\ f \downarrow \\ \mathbb{P}^1 \end{array}$ s.t. $-K_Y = f^*\mathcal{O}(1)$ and $|\mathrm{PsAut}(Y)| = \infty$.

[Coates - Corti - Galkin - Kasprzyk 2016, Cheltsov - Przyjalkowski 2018; Przyjalkowski 2018; Doran - Harder - Katzarkov - Ovcharenko - Przyjalkowski 2023]

105 deformation types of Fano 3-folds $- 7 - 6 = 92$ examples where $|\mathrm{PsAut}(Y)| = \infty$

Dolgachev: 2-reflexive $\Rightarrow$ $\mathrm{Mov}(Y)$ is not R.P.